VIRTUAL SETS ON AN ALGEBRAIC CURVE AS CONTRASTED WITH ABELIAN FUNCTION THEORY

BY

WILLIAM FRANKLIN ATCHISON

A.B., Georgetown College, 1938

M.A., University of Kentucky, 1940

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
MATHEMATICS IN THE GRADUATE SCHOOL OF
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PART I

A large range of theorems concerning the geometry on an algebraic curve has been proved by the use of Abelian integrals of the first kind. It is the purpose of this work to show how these Abelian parameters can be replaced to a large extent by the use of virtual sets of order zero. By a virtual set we mean the following. If on a curve C of genus p we have the equivalence

$$A_v + B'_{u'} \equiv A'_u + B_{v'}, \tag{1}$$

where A_v, $B_{v'}$, A'_u, $B'_{u'}$ are sets of points on C, the subscript indicating the number of points in the set, we may write

$$A_v - B_{v'} \equiv A'_u - B'_{u'} \tag{2}$$

even though the residual sets $A_v - B_{v'}$ and $A'_u - B'_{u'}$ do not exist. We say that the symbol $A_v - B_{v'}$ represents a virtual set of points on C, and that the virtual sets $A_v - B_{v'}$ and $A'_u - B'_{u'}$ are equivalent. The totality of virtual sets equivalent to a given virtual set is called a virtual complete linear series and is indicated by $|A_v - B_{v'}|$. The order of such a series $|A_v - B_{v'}|$ is the difference of the orders of the series $|A_v|$ and $|B_{v'}|$. Thus a virtual set of order zero is one in which the sets A_v and $B_{v'}$ contain the same number of points but are not equivalent sets. We shall be concerned only with virtual linear series of order zero. Dimension, however, has no significance in a virtual linear series. The virtual linear series $|A_v - B_{v'}|$ contains sets $A'_u - B'_{u'}$, where we may select A'_u arbitrarily as long as a set $B'_{u'}$ can be determined so that (1) holds. The series $|A'_u + B_{v'}|$ has dimension $(u + v') - p$ and thus to be able to find a $B'_{u'}$, we must have $(u + v') - p \geqslant v$, which puts no upper limit on the number of points which the set A'_u may contain.

A curve of genus zero contains no virtual complete linear

series of order zero, for on it any two sets of the same order are equivalent. Curves of genus $p > 0$ do. Select two non-equivalent sets of $v > 0$ points each, say A_v and B_v, and take their indicated difference $A_v - B_v$. For such a set we can always find an equivalent virtual set which is the difference of sets of order p, say $A_p - B_p$ so that we have

$$A_v - B_v \equiv A_p - B_p,$$

for if A_p is arbitrarily chosen the linear series $|A_p + B_v|$ has dimension v and the difference $|A_p + B_v| - A_v$ has residual sets $|B_p|$.

There are ∞^{2p} virtual differences

$$A_p - B_p \tag{3}$$

on a curve of genus p. However, there are not that many virtual complete linear series of order zero. This is seen by fixing a set of p non-special points P_p. Then for any virtual set (3) we may find a real set of points V_p such that

$$P_p - V_p \equiv A_p - B_p.$$

None of these sets can be equivalent for

$$P_p - V_p \equiv P_p - V_p'$$

implies that $V_p \equiv V_p'$ which implies their identity if V_p is not special. If V_p is special it is a member of a special g_1^p. Of these special (g_1^p)'s there are ∞^{p-2}. Hence, there are only ∞^p virtual complete linear series on the curve. These may be used as the parameters of the ∞^p sets of p points on C.

For fixed P_p and variable V_p we have in these virtual sets the p degrees of freedom which are implied in the well known Jacobi inversion problem by the use of which p-ads of points are determined when these points are used symmetrically as the upper limits for sums of integrals of the first kind. In the main the geometry on algebraic curves has been studied by the use of these sums. In particular applications sub-

multiples of the periods are employed. This work will be concerned with using these virtual sets to define sets which correspond to submultiples of the periods and thus enlarge the range of theorems which can be proved by algebraic rather than transcendental methods.

Finding these virtual sets involves a determination of the number of sets V_p such that $r(P_p - V_p) \equiv 0$ (r an integer > 1) but $P_p - V_p \not\equiv 0$. The complete linear series $|rP_p|$ is a g^{rp}_{rp-p}. By the use of the de Jonquières formula(1) (This formula is proved by algebraic methods, namely by the use of correspondence theory.) we see that in such a linear series it happens r^{2p} times that sets $rA_p^{(i)}$ are obtained, one of which is rP_p. Thus we find $r^{2p} - 1$ distinct virtual sets $P_p - A_p^{(i)}$ ($i = 1$ to $r^{2p} - 1$). These particular sets which in this algebra appear as divisors of zero will replace the submultiples of the periods as they appear in the transcendental expositions. Such virtual sets will be called subperiod sets of index r.

Using the properties of virtual linear series and mathematical induction the following theorem can be proved concerning these subperiod sets:

Theorem 1: There are $r^{2p}-1$ distinct virtual linear series of order zero, $u_i = |P_p - A_p^{(i)}|$, $i = 1$ to $r^{2p}-1$ on a curve of genus p with the property that $u_i \not\equiv 0$ but that $ru_i \equiv 0$. These can be expressed as $\sum_{i=1}^{2p} \lambda_i u_i$ for 2p properly chosen primitive u_i where $\lambda_i = 0, 1, \ldots, r-1$ and where not all $\lambda_i = 0$.

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(1) Jonquières, E de, "Mémoire sur les contacts multiples d'ordre quelconque des courbes de degré r, qui satisfont a des conditions donées, avec une courbe fixe du degré m; suivi de quelques réflexions sur la solution d'un grand nombre de questions concernant les propriétés projectives des courbes et des surfaces algébriques", Journal für die reine and angewandte Mathematik, Volume 66, (1866), Pages 289-321.

PART II

1. The Elliptic Cubic.

The case where p = 1 is essentially the case of the elliptic cubic. Many theorems concerning the geometry on the elliptic cubic are very easily proved by use of the elliptic parameter representation.(2) These theorems can be equally well proved by the use of virtual sets of order zero. Here there are ∞^1 virtual linear series of order zero represented by $|P - V|$ as the point V varies over the curve and P remains fixed. These are clearly in one to one correspondence with the points of the curve and thus may be taken as parameters on the cubic.

Take the fixed point P as a flex on the cubic C. In the g_2^3 cut out on C by the lines of the plane $3P \equiv L$, a line section and also $P_1 + P_2 + P_3 \equiv L$, where P_1, P_2, and P_3 are the points of intersection of any line with C. Subtracting the two above equivalences gives that

$$(P - P_1) + (P - P_2) + (P - P_3) \equiv 0.$$

Thus the condition that three points of C lie on a line is that the sum of their parameters be equivalent to zero, just as is the case for the elliptic representation in the canonical form. Similar relations hold for the intersections of curves of higher order with C.

The properties mentioned above coupled with the properties of the subperiod sets of index two and three enable one to prove essentially all the theorems cited in the reference given above.

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(2) Clebsch, A., "Leçons Sur La Géométrie", (1880), Volume II, Pages 355-393.

2. Systems of Contact Curves to a Given Curve of Genus p.

Given a set of p points P_p on a curve C of order n and genus p there is a complete linear series, a g^{rp}_{rp-p}, which contains the set rP_p and thus is determined by P_p. According to the de Jonquières formula cited previously this series contains r^{2p} such sets $rA^{(i)}_p$. The curves of the system $\sum$ which cut out this series pass through the sets $A^{(i)}_p$ r times and are the so-called contact curves of the system. $\sum$ may be taken to be a system of adjoint curves of order $m > n - 3$ on a fixed set of λ points on C other than the multiple points.

Clebsch(3) proved a sequence of theorems concerning contact curves by transcendental methods which can be proved by purely algebraic methods using virtual and real sets of points and applying the de Jonquières formula. The method will be illustrated by proving a few of the more important theorems.

Theorem 2: There exist r^{2p} distinct systems of contact adjoint curves of order $m > n - 3$, to a given curve C, having contacts of order r - 1 in p + u points of C where $ur \leq \lambda$. (Clebsch, op. cit., Page 234. In many cases some of these systems will be degenerate.)

Proof: When λ is large enough that an integer u can be selected such that $ur \leq \lambda$ we can take all the adjoint curves of order m on $\lambda - ur$ points yielding adjoints having contact of order r - 1 at p + u points. Each subperiod set of index r gives rise to a system of contact curves. Let Q_{p+u} be the points of contact of one such curve then there exists a set of points X_{p+u}, u of which may be selected arbitrarily such that

$$P_p - A^{(i)}_p \equiv Q_{p+u} - X_{p+u} \text{ or } P_p + X_{p+u} \equiv Q_{p+u} + A^{(i)}_p.$$

This gives a g^{2p+u}_{p+u} or a g^{p+u}_u in the case of the zero period.

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(3) Clebsch, A., "Leçons Sur La Géométrie", (1883), Volume III, Pages 229 - 252.

Hence all cases lead to a system of dimension u.

Theorem 3: The points of contact of r curves of contact of the same system are always situated on an adjoint curve of the m^{th} order which passes through the given fixed points. (Clebsch, op. cit., Page 234.)

Proof: Let r curves in the same system cut out the sets $B^{(i)}_{p+u}$ for i = 1 to r. If this is the system determined by $P_p - A^{(i)}_p$ then

$$P_p - A^{(i)}_p \equiv Q_{p+u} - B^{(i)}_{p+u} \qquad i = 1 \text{ to } r.$$

Adding these yields the equivalence,

$$0 \equiv rP_p - rA^{(i)}_p \equiv rQ_{p+u} - \sum_{i=1}^{r} B^{(i)}_{p+u}$$

or therefore

$$rQ_{p+u} \equiv \sum_{i=1}^{r} B^{(i)}_{p+u}.$$

Hence the set of B's is cut out by an adjoint of order m on the given fixed points.

Application of theorem 2 to the quartic of genus three yields that there are $2^6 = 64$ systems of conics simply tangent to the curve in four points, each system apparently of dimension one. The set Q_{p+u} used there may be taken as a line section L. This yields however that the system arising from the zero subperiod set gives rise to the set of lines of the plane counted twice, and this is a system of dimension two. This is the reason for saying that there are only 63 systems of proper contact conics. Theorem 3 yields that there is a conic on the eight points cut out on the quartic by any two contact conics of the same system.

Application of theorem 2 to the quartic again yields that there are 64 systems of cubics simply tangent to it in six points, each system being of dimension three, including the one coming from the zero subperiod set. Theorem 3 gives that the points of contact of any two cubics of the same system lie

on a cubic.

The 64 systems of contact cubics break up into 28 systems of one kind and 36 of another kind. Take all the conics on the points of contact, D_2, of a double tangent to the quartic. They form a g_3^6 with equivalent sets of points C_6. Now $2D_2 \equiv L$, a line section, and $D_2 + C_6 \equiv 2L$. Multiplying this second equivalence by two and subtracting the first from it gives $2C_6 \equiv 3L$. Hence on each of the sets of this g_3^6 there is a contact cubic and each must be in the same system since the sets of six are equivalent. Each of the 28 double tangents leads to such a g_3^6 and thus a system of contact cubics.

The above arguments are very readily extended to give the following general,

Theorem 4: There exist 63 systems of curves of order 2m and 64 systems of curves of order 2m + 1 simply tangent to a given quartic curve of genus three at the points where they intersect it. These 64 systems separate into two groups, one of 28 and the other of 36 systems. The points of contact of every curve of any one of the 28 systems are always situated on a curve of order m + 1 which passes through the points of contact of a given double tangent. Nothing similar happens for the other 36 systems. All possess the property that the points of contact of two curves of the same system are situated on a curve of the same order. (Clebsch, op. cit., Pages 250-251)

VITA

William Franklin Atchison was born in Smithfield, Kentucky, April 7, 1918, and received his elementary education there. For two years he attended high school in Ballardsville, Kentucky, and he was graduated from the LaGrange, Kentucky, High School in 1934. He received the degree of Bachelor of Arts (magna cum laude) from Georgetown College in 1938. During the school year of 1938-1939 he was enrolled in the mathematics department of the Graduate School of the University of Kentucky. The following school year he was a graduate assistant in the mathematics department and received the degree of Master of Arts in February, 1940. The title of his Master's thesis was "The Hessian". Since September, 1940, he has been a graduate assistant in the department of mathematics at the University of Illinois.

He is a member of Gamma Sigma Epsilon, Pi Mu Epsilon, the Society of Sigma Xi, and the American Mathematical Society.